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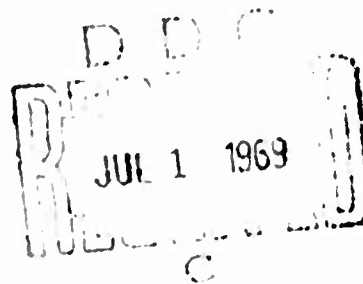
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# A System Debugging Model

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# A SYSTEM DEBUGGING MODEL*

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## SUMMARY

The problem treated here deals with the "debugging" of a new complex system during the initial period of its total life. During this period failures and errors are corrected as they occur, with resulting improvement in subsequent performance of the system. One mathematical idealization of this process leads to the assumption that system failure rate is decreasing with time. In practice, the debugging phase is considered completed when the failure rate reaches an equilibrium or constant value. Models are formulated for this phenomenon. Maximum likelihood estimates are obtained for relevant failure rate functions and for the end of the debugging period. A conservative upper confidence bound on the stable failure rate is obtained.

The problem is treated from a point of view which lies between a completely nonparametric approach in which no information is assumed available concerning the form of the distribution, and a parametric outlook in which the form of the distribution is assumed known but a finite number of parameters need to be estimated.

## 1. INTRODUCTION

It is common practice after installing a new complex system such as that involving a computer, airplane, etc., to "debug" it during the initial portion of its total life. During this debugging period, failures and errors are corrected as they occur, with resulting improvement in subsequent system performance. One mathematical idealization of this process leads to the assumption that system failure rate is decreasing with time. In practice, the debugging phase is considered completed when the failure rate reaches an equilibrium or constant value. An important problem is to determine when the constant failure rate condition has been achieved and to estimate the constant failure rate.

Another problem related to the debugging problem in many respects is the "burn-in" problem considered by Barlow, Madansky, Proschan and Scheuer (1968). The object of "burn-in" is to eliminate poor quality items in some population. However, in the "burn-in" problem considered there, items fail at most once and no repair occurs. Lewis (1964) developed a branching Poisson process for the analysis of computer failure patterns. Although he considers computer failure times which ostensibly occur after the debugging period, his model could be used for the debugging period as well. However, he makes more assumptions than we do in a highly structured mathematical model.

We obtain maximum likelihood estimators (MLE's) for the failure rate function and conservative confidence bounds on the failure rate at a specified time. This is done without the customary assumptions concerning the form of the life distribution. The approach is intermediate between a completely nonparametric point of view (in which no information is assumed available concerning the form of the distribution) and a parametric outlook (in which the form of the distribution is assumed known, but a finite number of parameters are to be estimated).

Next we find methods which allow us to claim with specified (high) assurance that the "stable" failure rate of a system which is being debugged during development and initial use is no greater than a certain value.

Clearly, without a knowledge of the form of the distribution of a relevant statistic, we cannot hope to obtain exact confidence bounds. However, we do obtain conservative confidence bounds. That is, our assurance is at least (instead of exactly equal to) a specified value that the reliability, failure rate, etc., falls in some confidence set determined from the observations. Of course, the price we pay is that the confidence sets tend to be larger than in the case in which the failure distribution is assumed to belong to a particular family of distributions. However, we shall show that the conservative confidence bounds obtained have the property that for a member of the class of distributions under consideration the confidence bounds are exact, not merely conservative

## 2. DEBUGGING MODELS

Suppose $X_1$, the time to the first failure, has distribution $F(t)$ with failure rate $r(t)$ which is nonincreasing for $t \geq 0$. After each failure, repair is performed in a negligible length of time so that the system operates again. Assume further that the system failure rate is restored to the value it had just prior to the failure. Specifically, assume that $X_i$, the time between the $(i-1)$st failure and the $i$-th failure has distribution

$$F_{X_i}(x) = \frac{F(S_{i-1} + x) - F(S_{i-1})}{\bar{F}(S_{i-1})} \quad \text{for } x \geq 0$$

(where $S_i = X_1 + \ldots + X_i$ and $\bar{F}(x) = 1-F(x)$), the conditional distribution of a system of age $S_{i-1}$.

Given observations $X_1, X_2, \ldots, X_n$, we wish to estimate $r(t)$ for $0 \leq t \leq S_n$. More generally, we may have k copies of the system, each copy independently operating as before. Observations $X_{ij}$, $i = 1, 2, \ldots, k$; $j = 1, \ldots, n_i$ are obtained. The distribution of $X_{ij}$ is

$$F_{X_{ij}}(x) = \frac{F(S_{i,j-1} + x) - F(S_{i,j-1})}{\bar{F}(S_{i,j-1})}$$

the conditional distribution of an item of age $S_{i,j-1} = X_{i1} + X_{i2} + \ldots + X_{i,j-1}$. Again we wish to estimate $r(t)$ during the observation period.

If the failure rate first decreases and then becomes constant for $t > t_o$, we may wish to estimate $t_o$ and $r(t_o)$. It is of interest to note the common-sense procedure often used in this situation to estimate $t_o$ and $r(t_o)$. A graph is drawn in which the comulative number of failures is plotted against elapsed operational time as in Fig. 1. (See Rosner (1961)). Debugging is terminated approximately at that point in time when the slopes

$$\frac{h - (h - 1)}{S_h - S_{h-1}} = \frac{1}{X_h}$$

of successive secants (shown by dashed lines) appear to have reached an equilibrium value. These slopes represent failure rates over successive time periods. System improvement corresponds to the situation in which the slopes, $\frac{1}{X_h}$, are decreasing with h. However, due to statistical fluctuations, some reversals will occur. The common-sense graphical procedure described above furnishes no precise way of taking into account these reversals. Our technique, based on maximum likelihood, provides for this.

In many practical situations it is not realistic to insist on determining the point $t_o$ beyond which failure rate is constant. Rather, for pragmatic purposes it suffices to find the point $t_1$, such that $r(t_1) - \lim_{t \to \infty} r(t) = \epsilon$ for some specified $\epsilon > 0$. Thus, we wish to find the point beyond which further reliability improvement can decrease the failure rate by only $\epsilon$. We wish to obtain maximum likelihood estimates of $t_1$ and a conservative upper bound on $r(t_1)$.

## 3. MAXIMUM LIKELIHOOD ESTIMATES FOR DEBUGGING MODELS

We begin by giving the MLE, $\overset{*}{r}_n(t)$, for a decreasing failure rate function based on a sample of size n: $X_1, X_2, \ldots, X_n$ from one copy of the system. The derivation of the MLE is very similar to that given for a decreasing failure rate function based on n independent observations from a DFR distribution (Marshall and Proschan (1965)) and is therefore omitted. (See also Brunk (1965) for related estimates.)
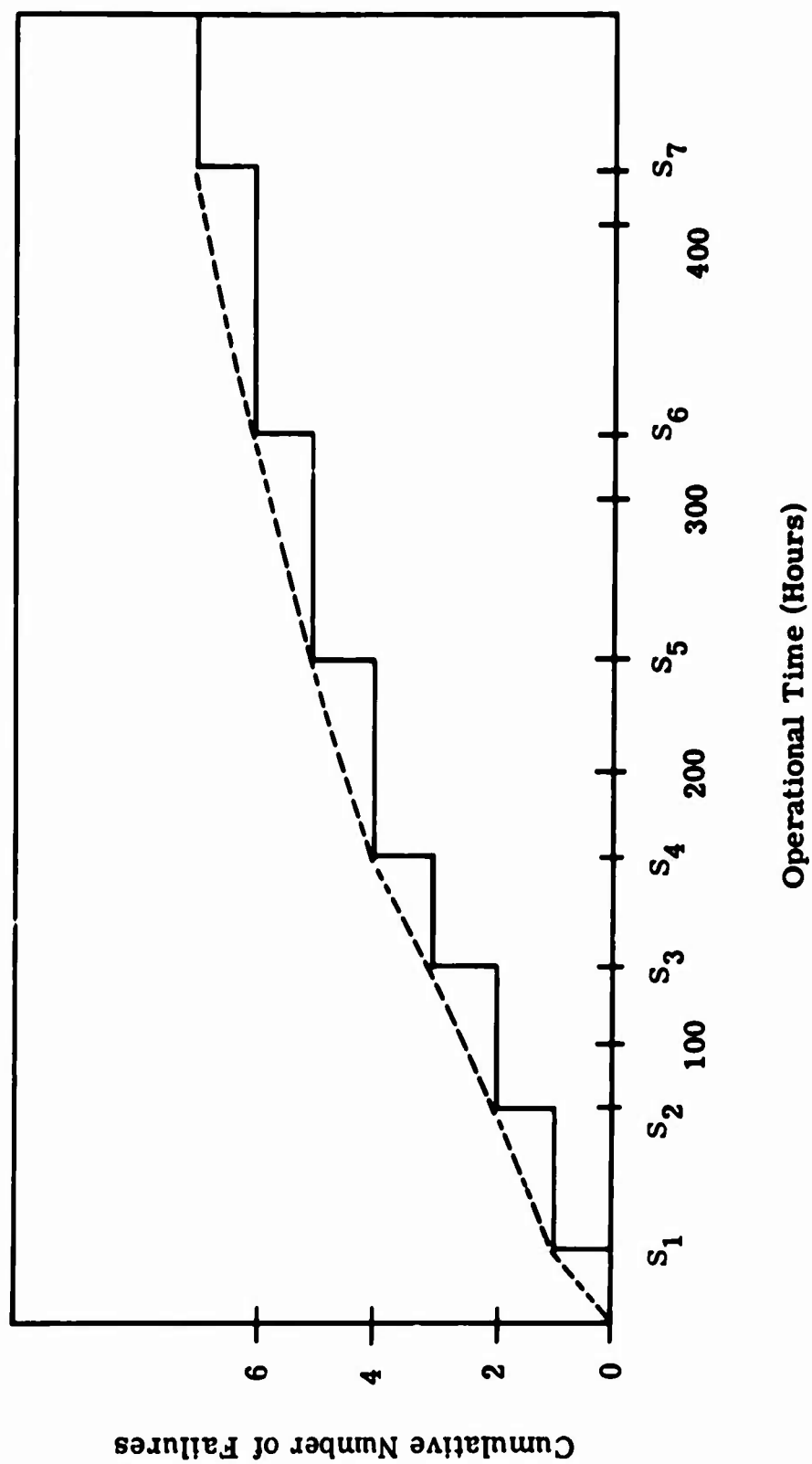
Fig. 1 - Cumulative number of failures versus time

Recall the notation $S_i = X_1 + X_2 + \ldots + X_i$ with the convention that $S_0 = 0$. The MLE $\hat{r}_n(t)$ is constant on the intervals $(S_i, S_{i+1}]$ for $i = 0,\ldots, n-1$. The MLE for $r(t)$ on $(S_i, S_{i+1}]$ is $X_{i+1}^{-1}$ before taking account of the fact that the distribution is DFR. If it turns out that $X_1^{-1} \geq X_2^{-1} \geq \ldots \geq X_n^{-1}$, then we conclude that $\hat{r}_n(t) = X_{i+1}^{-1}$ for $S_i < t \leq S_{i+1}$, $i = 0, 1,\ldots, n-1$. If a reversal occurs, say $X_{i+1}^{-1} < X_{i+2}^{-1}$, then we must average to obtain a common estimate of failure rate, $\left\{ \frac{1}{2} (X_{i+1} + X_{i+2}) \right\}^{-1}$, for $S_i < t \leq S_{i+2}$. Next we examine $X_1^{-1}, \ldots, X_i^{-1}, \left\{ \frac{1}{2} (X_{i+1} + X_{i+2}) \right\}^{-1}, \left\{ \frac{1}{2} (X_{i+1} + X_{i+2}) \right\}^{-1}, X_{i+3}^{-1}, \ldots, X_n^{-1}$ to see if these estimates of the failure rate on the successive intervals are decreasing. If so, they constitute the MLE's of the failure rates on the successive intervals. If not, we continue to average until no reversals remain. At the end of this process, we obtain MLE's $r_{1,n_1} \geq r_{n_1+1,n_2} \geq \ldots \geq r_{n_k+1,n}$ satisfying:

$$r_{1,n_1} = \left\{ \frac{1}{n_1} (X_1 + \ldots + X_{n_1}) \right\}^{-1},$$

$$r_{n_1+1,n_2} = \left\{ \frac{1}{n_2 - n_1} (X_{n_1+1} + \ldots + X_{n_2}) \right\}^{-1},$$

$$\vdots$$

$$r_{n_k+1,n} = \left\{ \frac{1}{n - n_k} (X_{n_k+1} + \ldots + X_n) \right\}^{-1},$$

and

$$\hat{r}_n(t) = \begin{cases} r_{1,n_1} & \text{for } 0 \leq t \leq S_{n_1} \\ r_{n_1+1,n_2} & \text{for } S_{n_1} < t \leq S_{n_2} \\ \vdots & \\ r_{n_k+1,n} & \text{for } S_{n_k} < t \leq S_n. \end{cases} \qquad (3.1)$$

No estimate of $r(t)$ is made for $t > S_n$ since no data are available for that time interval.

Example 1. In Fig. 1 the cumulative number of failures versus times of failure is graphed for the following data

| Time of Failure | Time Between Successive Failures |
|---|---|
| $S_1 = 25$ hours | $X_1 = 25$ hours |
| $S_2 = 75$ hours | $X_2 = 50$ hours |
| $S_3 = 125$ hours | $X_3 = 50$ hours |
| $S_4 = 165$ hours | $X_4 = 40$ hours |
| $S_5 = 240$ hours | $X_5 = 75$ hours |
| $S_6 = 310$ hours | $X_6 = 70$ hours |
| $S_7 = 410$ hours | $X_7 = 100$ hours |

We are assuming that $r(t)$ is decreasing in t. If there were no reversals in the observed failure rates on successive intervals, the estimate of $r(t)$ would be

$$\hat{r}(t) = \frac{1}{X_i}, \quad S_{i-1} < t \leq S_i, \quad i = 1, \ldots, 7;$$

i.e.,

$$\frac{1}{25}; \frac{1}{50}; \frac{1}{50}; \frac{1}{40}; \frac{1}{75}; \frac{1}{70}; \frac{1}{100} .$$

However, since $\frac{1}{50} < \frac{1}{40}$ and $\frac{1}{75} < \frac{1}{70}$, we have two reversals.

By combining the second, third, and fourth estimates (adding numerators of the three estimates to obtain a new numerator, and adding denominators to obtain a new denominator), we obtain as our new, tentative estimate of $r(t)$:

$$\frac{1}{25}; \frac{3}{140}; \frac{3}{140}; \frac{3}{140}; \frac{1}{75}; \frac{1}{70}; \frac{1}{100} .$$

The reversal $\frac{1}{75} < \frac{1}{70}$ is left. Combining these as before, we obtain finally as the MLE of r at the observations:

$$\hat{r}(25) = \frac{1}{25}$$

$$\hat{r}(75) = \hat{r}(125) = \hat{r}(165) = \frac{3}{140}$$

$$\hat{r}(240) = \hat{r}(310) = \frac{2}{145}$$

$$\hat{r}(410) = \frac{1}{100} \; .$$

Between successive observations, $\hat{r}$ is, of course, constant. Using this "smoothed" data, we obtain a new graph in Fig. 2, in which the slopes (failure rates) of Fig. 1 are smoothed.

## MLE for a Decreasing Failure Rate from k Copies of the System

Using the same techniques as in the case of a single copy of the system, treated above, we may obtain the MLE of the failure rate, assumed decreasing. Again the derivation of the MLE is similar to that given in Marshall and Proschan (1965) and is omitted.

First the actual failure times (not intervals between failures) for all k systems are pooled and ordered. Call these ordered observations $T_1 \leq T_2 \leq \ldots \leq T_n$, where $n = \sum_{i=1}^{k} n_i$. Between successive $T_i$, the failure rate estimate is constant as above. Our initial estimate of the failure rate in an interval, before imposing the constraint that the failure rate be decreasing, is computed as the reciprocal of the total test time observed in that interval. Thus, on $[0, T_1]$, the initial estimate is $(n \, T_1)^{-1}$, on $(T_1, T_2]$, the initial estimate is $\{N_1(T_2-T_1)\}^{-1}$, on $(T_2, T_3]$, the initial estimate is $\{N_2(T_3-T_2)\}^{-1}$, ..., on $(T_{n-1}, T_n]$, the initial estimate is $(T_n - T_{n-1})^{-1}$ where $N_i$ is the number of systems simultaneously in operation during $(T_i, T_{i+1}]$. On $(T_n, \infty)$, no estimate of failure rate is made since no failures are observed.

The initial estimates are then compared; if they are in decreasing order, they constitute a MLE of $r(t)$ on $[0, T_n]$. If a reversal occurs, we average as above to eliminate it. For example, if $N_{i-1}(T_i - T_{i-1}) > N_i(T_{i+1} - T_i)$, the revised estimate of the failure rate on $(T_{i-1}, T_{i+1}]$ is $\left\{\frac{1}{2} [N_{i-1}(T_i-T_{i-1}) + N_i(T_{i+1}-T_i)]\right\}^{-1}$. We continue averaging in this

fashion until all reversals are eliminated. The resulting estimate is the MLE of $r(t)$ on $[0, T_n]$ under the assumption that $r(t)$ is a decreasing function.

## Strong Consistency of the MLE

In this subsection we show that the MLE for the decreasing failure rate obtained above from k copies of the system is consistent. We assume that at least t hours of operation are observed for each of k systems. We shall show that as $k \to \infty$, $\hat{r}_k(t) \to r(t)$ with probability one.

Let $n_i(t)$ be the number of failures of the i-th system and $n = n(t) = \sum_{i=1}^{k} n_i(t)$ be the total number of failures over all k copies in $[0, t]$.

Then

$$P[n_i(t) = j] = \frac{[\int_o^t r(x)\ dx]^j}{j!}\ e^{-\int_o^t r(x)\ dx},$$

$\{n_i(t),\ t \geq 0\}$ is a non-homogenous Poisson process with intensity function $r(t)$, and $En_i(t) = \int_o^t r(x)\ dx$.

**Lemma 3.1.** Suppose k systems operate between observed failure times $T_{i-1}$ and $T_i$. Then $Z_i \equiv k \int_{T_{i-1}}^{T_i} r(u)\ du$, $i = 1,2,\ldots,n$, are independently and identically distributed with exponential distribution $P[Z_i \leq z] = 1-e^{-z}$.

For a proof see Barlow and Proschan (1969). ||

Let $T_i \leq t_o < T_{i+1}$ and $\hat{r}_k(t_o) = \sup_{t \geq i+1}\ \inf_{s \leq i}\ \dfrac{t-s}{\sum_{j=s}^{t-1} k(T_{j+1}-T_j)}$

**Theorem 3.2.** Let $r(x)$ be nonincreasing in $x \geq 0$ and each of k systems be observed for at least $t_o$ hours of operation. Then

$$r(t_o^-) \geq \lim_{k \to \infty} \sup \hat{r}_k(t_o) \geq \lim_{k \to \infty} \inf \hat{r}_k(t_o) > r(t_o^+)$$

with probability one.

**Proof.** Since the proof is essentially the same as in Marshall an Proschan (1965) (abbreviated MP(1965)), we merely indicate the necessary

changes in certain details. To make the identifications easier, we assume that $r(x)$ is <u>nondecreasing</u> rather than <u>nonincreasing</u> as in the hypothesis above. This merely affects the ordering of the inequalities and the order in which we maximize and minimize.

The $X_i$ in MP(1965) become $T_i$ in our model, while the number, n-1, of items on test between $X_i$ and $X_{i+1}$ becomes the number k of items on test between $T_i$ and $T_{i+1}$. Let $a_j(n) + 1$ be the index of the largest $T_i$ observation $\leq t_j$, j=0,1, as in MP(1965). Then

$$Z_i = k \int_{T_i}^{T_{i+1}} \frac{r(u)}{r(t_1)} \, du$$

are independent, identically distributed random variables with mean $\frac{1}{r(t_1)}$. As in MP(1965), we wish to show that $P[\lim \sup B_n] = 0$, where

$$B_n = \left[ \max_{a_1(n)-a_0(n) \leq m \leq a_1(n)} \left| m^{-1} \sum_{i=1}^{m} (Z_i - \frac{1}{r(t_1)}) \right| \geq \epsilon \right].$$

The main change in the proof occurs with respect to our definition of $A_n$. Their set $A_n$ becomes

$$A_n = [ |a_i(n) - k \int_0^{t_1} r(x) \, dx| < k \, \delta, \ i=0,1],$$

where $\delta$ satisfies $0 < 2\delta < \int_{t_0}^{t_1} r(x) \, dx$. Since

$$\frac{a_i(n)}{k} = \frac{n_1(t_i)+...+ n_k(t_i)}{k} \xrightarrow{a.s} \int_0^{t_1} r(x) \, dx$$

by the strong law of large numbers for i=0,1, we have $P[\lim \sup A_n^c] = 0$, and the remainder of the proof is identical with that of MP(1965).
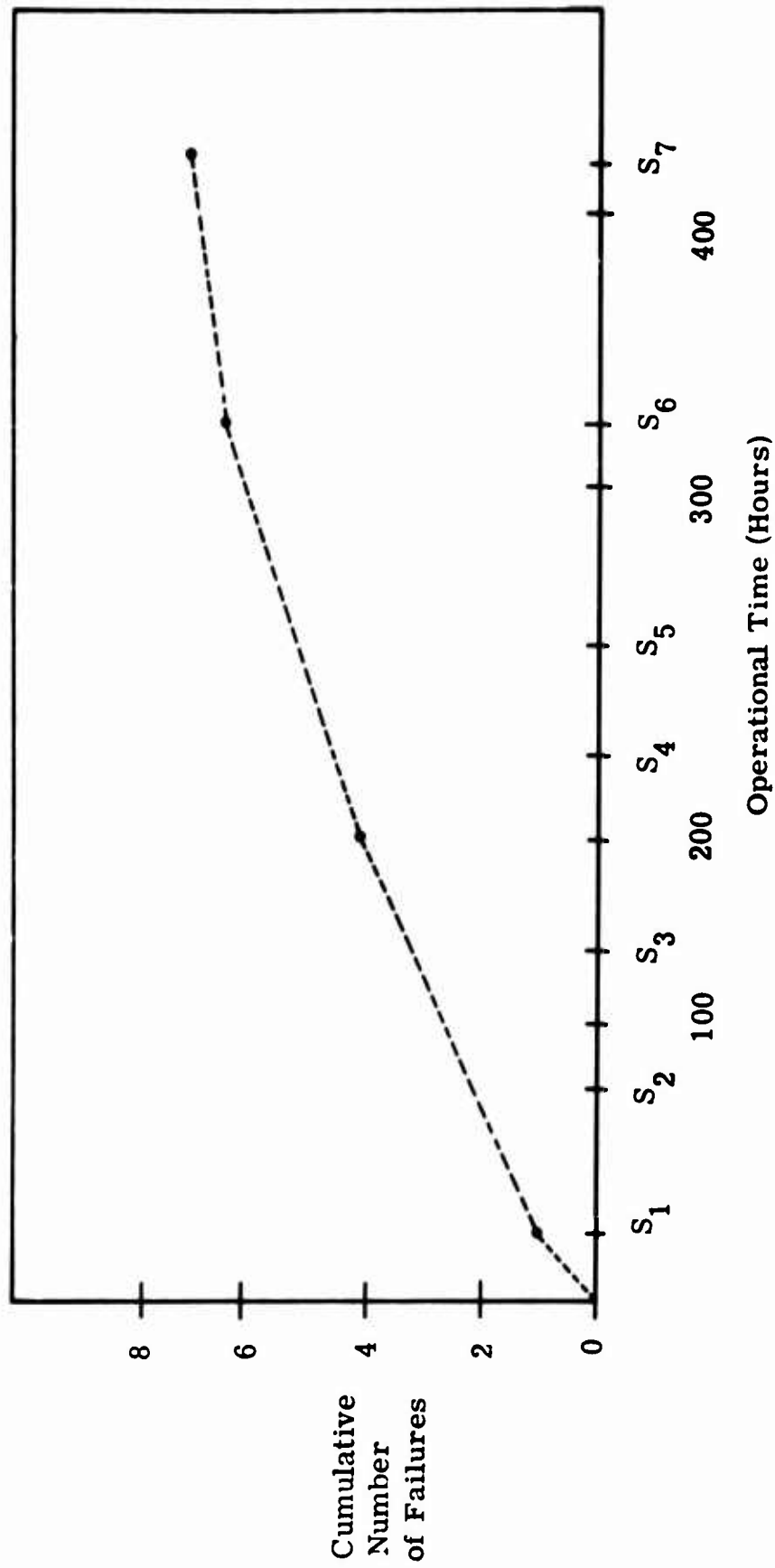
Fig. 2 – "Smoothed" graph of cumulative number of failures versus time

(Note: See Fig. 1 for unsmoothed graph.)

Example 2: The method for obtaining the MLE of $r(t)$ in the system debugging model is illustrated for two copies of the same system. Suppose System 1 fails at times

$$S_{11} = 25 \text{ hrs.}, \quad S_{12} = 125, \quad S_{13} = 240,$$

and that System 2 fails at times

$$S_{21} = 75, \quad S_{22} = 165, \quad S_{23} = 310, \quad S_{24} = 410.$$

If the failure times are pooled and are denoted by $T_1$, $T_2$,..., as before, then our estimate of $r(t)$, assuming no reversal, would be

$$\hat{r}(t) = \begin{cases} \dfrac{1}{2(T_i - T_{i-1})} & \text{for } T_{i-1} < t \le T_i, \quad i \le 5 \\[2ex] \dfrac{1}{T_i - T_{i-1}} & \text{for } T_{i-1} < t \le T_i, \quad i > 5; \end{cases}$$

i.e., $\dfrac{1}{50}$; $\dfrac{1}{100}$; $\dfrac{1}{100}$; $\dfrac{1}{80}$; $\dfrac{1}{150}$; $\dfrac{1}{70}$; $\dfrac{1}{100}$. Since $\dfrac{1}{100} < \dfrac{1}{80}$ and $\dfrac{1}{150} < \dfrac{1}{70}$, we have two reversals. By combining the second, third, and fourth estimates as before, we obtain

$$\frac{1}{50}; \quad \frac{3}{280}; \quad \frac{1}{150}; \quad \frac{1}{70}; \quad \frac{1}{100}.$$

By combining the estimates $\dfrac{1}{150}$ and $\dfrac{1}{70}$, which represent a reversal, we obtain

$$\frac{1}{50}; \quad \frac{3}{280}; \quad \frac{2}{220}; \quad \frac{1}{100}.$$

The reversal $\dfrac{2}{220} < \dfrac{1}{100}$ is still left. Combining as before, we finally obtain

$$\hat{r}(t) = \begin{cases} \dfrac{1}{50} & \text{for } 0 \le t \le 25 \\[2ex] \dfrac{3}{280} & \text{for } 25 < t \le 165 \\[2ex] \dfrac{3}{320} & \text{for } 165 < t \le 410. \end{cases}$$

## MLE for the Time Debugging Ends

To estimate the end of the debugging period, we first compute the MLE for the failure rate as above. Now suppose $r(t)$ is decreasing for $0 \leq t \leq t_o$ and constant for $\cdot \geq t_o$. The MLE for the end point of the debugging period is the beginning of the last averaging interval. This is a consequence of the invariance property of the MLE; i.e., the MLE of a function $u(r(t))$ of the failure rate is the same function $u(\hat{r}(t))$ of the failure rate MLE. In the notation of (3.1), $\hat{t}_o = S_{n_k}$. The MLE for $r(t_o)$ is the value of $\hat{r}(t)$ in the last interval, that is $\hat{r}(t_o) = r_{n_k+1,n}$. It turns out that in this case the MLE of $t_o$ is a poor one (i.e., is not even consistent) since, as the number of observations increases to infinity, $t_o$ converges almost surely to infinity, as may be shown by using the result of Andersen (1954), p. 218, top.

Now suppose that we wish to estimate $t_1$, the left-most point beyond which the failure rate does not decrease by more than $\epsilon$. Specifically, $t_1 = \inf\{t: r(t) - \lim_{t \to \infty} r(t) \leq \epsilon\}$. Denote by $k^*$ the smallest index k such that $\hat{r}(S_k) - \hat{r}(S_n) \leq \epsilon$. Then the MLE for $t_1$ is $\hat{t}_1 = S_{k^*}$. Again we are using the invariance property of the MLE described above.

Since $\hat{r}(t)$ is a consistent estimator of $r(t)$, it follows readily that $\hat{t}_1$ is a consistent estimator of $t_1$, assuming $r(t)$ is strictly decreasing in any interval containing $t_1$.

## 4. CONSERVATIVE CONFIDENCE BOUNDS

In this section methods are presented which allow us to claim with specified (high) assurance that the "stable" failure rate of a system which is being debugged during development and initial use is no greater than a certain value.

The basic idea in obtaining the conservative confidence bound on $r_o$, the stable failure rate, may be stated intuitively as follows. The observation, $X_i$, is a random variable from a distribution whose failure rate at each point of time is at least as great as $r_o$, $i = 1,\ldots,n$. Therefore, if one uses observations $X_1, X_2, \ldots, X_n$ to estimate a single failure rate (pretending that all the $X_i$ are from a common exponential distribution), the estimate will tend to be higher than $r_o$. Similarly, an upper confidence bound for this common failure rate, calculated from

the observations $X_1$, $X_2$, ..., $X_n$ as though they were a sample from a single exponential distribution, will constitute a conservative upper confidence bound for $r_o$. We make these ideas precise now.

Lemma 4.1. Let $X_1$ have distribution $F(x)$, $X_2$ have conditional distribution

$$\frac{F(X_1+x) - F(X_1)}{\bar{F}(X_1)} , \ldots,$$

$X_i$ have conditional distribution

$$\frac{F(S_{i-1}+x) - F(S_{i-1})}{\bar{F}(S_{i-1})} , \ldots,$$

where F has failure rate $r(t) \geq r_o$ for all $t \geq 0$. Let $Y_1$, $Y_2$, ...,$Y_n$ be independent observations from the exponential distribution with failure rate $r_o$. Then $\sum_1^n X_i$ is stochastically smaller[*] than $\sum_1^n Y_i$.

Proof. First assume that F is continuous. Let the random variables $X_1$, $X_1 + X_2$, ..., $\sum_1^n X_i$ be simultaneously transformed into random variables $Y_1'$, $Y_1' + Y_2'$, ..., $\sum_1^n Y_i'$ under the transformation

(4.1)   $Y_1' + \ldots + Y_i' = - \frac{1}{r_o} \log \bar{F}(X_1 + \ldots + X_i)$,  $i = 1, \ldots, n$.

Then for $i = 1, \ldots, n$,

$$P[Y_i' > u] = P[- \frac{1}{r_o} \log \bar{F}(X_1 + \ldots + X_i) + \frac{1}{r_o} \log \bar{F}(X_1 + \ldots + X_{i-1}) > u]$$

$$= P \left[ \log \frac{\bar{F}(X_1+\ldots+X_i)}{\bar{F}(X_1+\ldots+X_{i-1})} < - r_o u \right]$$

$$= P \left[ \frac{\bar{F}(X_1+\ldots+X_i)}{\bar{F}(X_1+\ldots+X_{i-1})} < e^{-r_o u} \right] = e^{-r_o u},$$

since the random variable

$$\frac{\bar{F}(X_1+\ldots+X_i)}{\bar{F}(X_1+\ldots+X_{i-1})}$$

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[*]The random variable U is said to be stochastically smaller than the random variable V if $P(U \geq t) \leq P(V \geq t)$ for each real value t.

is uniformly distributed on $[0,1]$.[*] Thus, the $Y_1', \ldots, Y_n'$ are independently distributed according to

$$G_{r_o}(x) = 1 - e^{-r_o x},$$

the exponential distribution with failure rate $r_o$.

Next observe that if $y = -\frac{1}{r_o} \log \bar{F}(x)$, then

$$\frac{dy}{dx} = \frac{r(x)}{r_o} \geq 1 \text{ for all } x \geq 0.$$

Thus, under the transformation (4.1)

$$Y_1' + \ldots + Y_n' \geq X_1 + \ldots + X_n.$$

It follows from Lehmann (1959), Lemma 1, p. 73,[**] that $\sum_1^n Y_i$ is stochastically larger than $\sum_1^n X_i$.

If $F$ is not continuous, the same result may be obtained by limiting arguments.‖

We now apply Lemma 4.1 to obtain a conservative confidence bound on $r_o$ from observations $X_1, \ldots, X_n$.

Since $Y_1, \ldots, Y_n$ are exponential with failure rate $r_o$, $\chi^2_{1-\alpha}(2n)/2\sum_1^n Y_i$ is an upper $100(1-\alpha)$ percent confidence bound on $r_o$, where $\chi^2_{1-\alpha}(2n)$ is the $100(1-\alpha)$ percentile of the chi-square distribution with $2n$ degrees of freedom. Hence

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[*] If a random variable $T$ has survival probability function $\bar{H}$, a continuous function, then the random variable $\bar{H}(T)$ is uniformly distributed on $[0,1]$; $\bar{F}(S_{i-1} + x)/\bar{F}(S_{i-1})$ is the conditional survival probability function of $X_i$ given $S_{i-1}$.

[**] This Lemma states: "Let $F_0$ and $F_1$ be two cumulative distribution functions on the real line. Then $F_1(x) \leq F_0(x)$ for all $x$ if and only if there exist two nondecreasing functions $f_0$ and $f_1$, and a random variable $V$, such that (a) $f_0(v) \leq f_1(v)$ for all $v$; and (b) the distributions of $f_0(V)$ and $f_1(V)$ are $F_0$ and $F_1$ respectively." In our case take

$$f_0(v) = v, \quad f_1(v) = -\frac{1}{r_o} \log \bar{F}(v), \quad F_0 = F, \quad F_1 = G_{r_o}.$$

$$1-\alpha = P[r_o \leq \chi^2_{1-\alpha}(2n)/2\Sigma^n_1 Y_i] \leq P[r_o \leq \chi^2_{1-\alpha}(2n)/2\Sigma^n_1 X_i].$$

Thus $\chi^2_{1-\alpha}(2n)/2\Sigma^n_1 X_i$ is a conservative $100(1-\alpha)$ percent upper confidence bound on $r_o$. Note that if F is the exponential distribution, the confidence bound is exact.

Examples

(i) In Example 1, $n = 7$ and $\Sigma X_i = 410$. Choosing $\alpha = .05$, we find from chi-square tables that $\chi^2_{.95}(14) = 23.7$. Thus, a conservative 95 percent upper confidence bound on $r_o$ is $23.7/820 = .0289$. That is, $P[r_o \leq .0289] \geq .95$.

(ii) The data in Example 2 come from two copies of the system. The procedure for more than one copy of the system is essentially the same as that for one copy of the system, viz. a conservative $100(1-\alpha)$ percent upper confidence bound on $r_o$ is $\chi^2_{1-\alpha}(2n)/2\Sigma^k_1 \Sigma^{n_i}_1 X_{ij}$, where $n = \Sigma^k_1 n_i$, k is the number of copies, and $n_i$ is the number of observations on the i-th copy. For the data in Example 2, $k = 2$, $n_1 = 3$, $n_2 = 4$ (so that $n = 7$), and $\Sigma\Sigma X_{ij} = 650$. Choosing $\alpha = .05$, a conservative 95 percent upper confidence bound on $r_o$ is $23.7/1300 = .018$. That is, $P[r_o \leq .018] \geq .95$.

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